Homework 7

This assignment is due by **11:59pm on Friday 11/1** on gradescope. You should start a new page for each problem (if a problem has multiple parts, the parts can be completed on the same page). After completing the assignment, save your solutions as a single pdf and upload it to gradescope. Be sure to **assign your pages** on the gradescope upload page; failure to do so will result in a penalty.

You may collaborate with your peers on homework assignments and discuss points of confusion with the instructor and/or TAs. However, you must write your own solutions independently and you may not look at or copy any one else's solutions. You must show all of your work to receive full credit (with the exception of row operations).

1. On the 10/24 Recitation Worksheet, you showed that the 2×2 determinant formula satisfies the first two determinant axioms. Now show the formula satisfies the third determinant axiom (linearity in each row).

2. Recall that if A is a 2×2 matrix, then $|\det A|$ is the area of the parallelogram formed by the columns of A and if A is 3×3, the $|\det A|$ is the volume of the parallelopiped formed by the columns of A. More generally, if A is $n \times n$, where $n \geq 4$, then $|\det A|$ is called the "hypervolume" of the $n-$dimensional region formed by the columns of A.
   (a) Use determinants to find the area of the triangle with corners at the coordinate points $(3, 4)$, $(2, 1)$, and $(0, 0)$.
   (b) Consider the $4-$dimensional region H given by the vectors
   $$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$
   Use determinants to find the hypervolume of H. Explain your steps (you may use row reduction or cofactor expansions).
   (c) Show that the vectors in part (b) are orthogonal and have the same length. Such an object is called a *hypercube* (a 4-dimensional analog of a cube (with is a 3-dimensional analog of a square)). Analogous to the area formula of a square or the volume formula of a cube, the volume of a hypercube with side length s is given by s^4; use this formula to verify your answer to part (b).

3. Use cofactor expansions to find the determinant of the following matrix. Be sure to show your work and explain your steps.
   $$A = \begin{bmatrix} -2 & 1 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 1 & -2 & 4 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

4. The cofactor formula of the inverse is not typically used for computational purposes (as it is much less efficient than row reduction). Rather, it is useful for proving general facts about inverses. Use the cofactor formula for inverses to prove that if A is an invertible 3×3 upper triangular matrix, then A^{-1} is also upper triangular.

   [Note: this fact can similarly be proved to be true for any $n \times n$ matrix]

5. Let $Q_{\frac{\pi}{4}}$ and $R_{\frac{\pi}{4}}$ be the 2×2 rotation and reflection matrices defined on Homework 6. Determine the eigenvalues of these two matrices geometrically; if there are no eigenvalues, explain why. You shouldn't need to compute anything for this problem.

6. Let P be an $n \times n$ projection matrix that projects vectors onto a subspace W of $\mathbb{R}^n$.
   (a) Algebraically prove that if λ is an eigenvalue of P then $\lambda \in \{0, 1\}$
   (Hint: recall from Homework 5 that $P^2 = P$).
   (b) Suppose $0 < \dim W < n$. Geometrically show that 0 and 1 are both eigenvalues of P and determine (geometrically) the corresponding eigenspaces.

7. Consider the Markov matrix $M = \begin{bmatrix} .2 & 0 & 0 \\ 0 & .2 & 0 \\ .8 & .8 & 1 \end{bmatrix}$.
   Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{x}_0 = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$.
   (a) Show that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of M and find the corresponding eigenvalues.
   (b) Set up and solve a system in order to express $\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.
   (c) Use part (b) to determine a formula for $\mathbf{x}_k = M^k \mathbf{x_0}$ in terms of k.
   (d) Determine what the Markov chain defined in part (c) converges to by computing its limit.